8. BURNETT D., The distribution of velocities in a slightly non-uniform gas. Proc. Lond. Math. Soc., 39, 5/6, 1935.
9. SHAVALIYEV M.SH., Transfer phenomena in the Barnett approximation in multicomponent gaseous mixtures. Izv. Akad. Nauk SSSR, Magn. Gidrodinamika, 1, 1974.
10. PRIGOZHIN I., Non-equilibrium Statistical Mechanics, Mir, Moscow, 1964.
11. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics. 6, Hydrodynamics. Nauka, Moscow, 1986.
12. STRUMINSKII V.V. and KUROCHKIN V.I., The kinetic theory of dense gases, Dokl. Akad. Nauk SSSR, 257, 1, 1981.
13. STRUMINSKII V.V., A general theory of finely dispersed media, in: The Mechanics of Multicomponent Media in Technoligical Processes, Nauka, Moscow, 1977.

Translated by R.C.G.

PMM U.S.S.R.,Vol.50,No.6,pp.711-718,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON THE STABILITY OF THE STEADY-STATE MOTIONS OF SYSTEMS WITH QUASICYCLIC COORDINATES*

V.V. RUMYANTSEV

The stability of the steady-state motions of a system with quasicyclic coordinates under the action of potential and dissipative forces and also forces which depend on the quasicyclic velocities is investigated. The results are applied to the problem of the stability of the steady-state plane-parallel motions of a rotor on a shaft which is set up in elasticated bearings with a non-linear reaction /1/.

The stability of the stationary motions and relative equilibria of systems with a single cyclic (quasicyclic) coordinate has previously been investigated /2/ from a common point of view. The question of the stability of the stationary motions of systems with quasicyclic coordinates under the action of constant and dissipative forces has been considered in /3/. The results obtained in /2/ have been generalized /4/ to systems with several cyclic (quasicyclic) coordinates and, additionally, a third regime of uniform motions, which includes the regime considered in /3/, has also been investigated.

1. Let us consider a holonomic mechanical system which is characterized by a Lagrange function $L = L(q_i, q_i^{\cdot}, \varphi_j^{\cdot})$, where $q_i\ (i = 1, \ldots, k), \varphi_j\ (j = k+1, \ldots, n)$ are generalized coordinates and $q_i^{\cdot} \equiv dq_i/dt$, $\varphi_j^{\cdot} \equiv d\varphi_j/dt$ are the generalized velocities of the system and, moreover, the function L is explicitly independent of the coordinates φ_j and the time t. Such a system with cyclic coordinates φ_j may execute stationary motions

$$q_i = q_{i0},\quad q_i^{\cdot} = 0;\quad \varphi_j^{\cdot} = \omega_j,\quad \varphi_j = \omega_j t + \varphi_{j0} \tag{1.1}$$

in which the positional coordinates q_i and the cyclic velocities $\varphi_j^{\cdot}$ remain constant over the whole time of the motion. At the same time the constants ω_j are either specified arbitrarily within certain limits and the constants q_{i0} are determined from the equations

$$\partial L/\partial q_i = 0 \tag{1.2}$$

in which it follows to put $q_i^{\cdot} = 0, \varphi_j^{\cdot} = \omega_j$ or q_{i0} and ω_j are determined from Eqs.(1.2) and the first integrals of the equations of motion

$$\partial L/\partial \varphi_j^{\cdot} = c_j \tag{1.3}$$

in which follows to put $q_i^{\cdot} = 0; c_j$ are arbitrary constants of integration. Here and everywhere subsequently

$$i, l = 1, \ldots, k;\quad j, s = k+1, \ldots, n.$$

*Prikl.Matem.Mekhan.,50,6,918-927,1986

unless something to the contrary is indicated.

Let us consider the case when the generalized potential forces $Q_i(q, q^\cdot, \varphi^\cdot)$ and $\Phi_j(q, q^\cdot, \varphi^\cdot)$ act on the system. In the case when $\Phi_j \not\equiv 0$, the coordinates φ_j are called quasicyclic coordinates. A system with quasicyclic coordinates may also execute motions of the form (1.1) if the coditions

$$\partial L/\partial q_i + Q_i = 0, \quad \Phi_j = 0 \tag{1.4}$$

are satisfied when $q_i^\cdot = 0$.

The values of the constants q_{i0} and ω_j which satisfy conditions (1.4), generally speaking, will differ from their values in the stationary motion of a system with cyclic coordinates only if the conditions

$$Q_i(q_{i0}, 0, \omega_j) = 0, \quad \Phi_j(q_{i0}, 0, \omega_s) = 0 \tag{1.5}$$

are not satisfied for them.

Several cases of (1.5) are considered in /4/.

As a rule, in the case of a system with quasicyclic coordinates subjected to specified generalized forces, the constants ω_j as well as the constants q_{i0} are determined from Eqs. (1.4) which we shall subsequently assume to be solvable.

It is convenient /4/ to pass from the variables φ_j to the variables

$$\xi_j = \varphi_j - \omega_j t \tag{1.6}$$

After substitution of the variables (1.6), the Lagrange function (we shall retain the previous notation for it) will have the structure

$$L(q, q^\cdot, \xi^\cdot) = L_2(q, q^\cdot, \xi^\cdot) + L_1(q, q^\cdot, \xi^\cdot) + L_0(q)$$

where $L_\alpha(q, q^\cdot, \xi^\cdot)$ are the homogeneous powers $\alpha = 0, 1, 2$ of the form of the variables $q_i^\cdot$, $\xi_j^\cdot$.

The Lagrange equations of motion of the system have the form

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^\cdot} - \frac{\partial L}{\partial q_i} = Q_i, \quad \frac{d}{dt}\frac{\partial L}{\partial \xi_j^\cdot} = \Phi_j \tag{1.7}$$

The solutions of Eqs.(1.7)

$$q_i = q_{i0}, \quad q_i^\cdot = \xi_j^\cdot = 0 \tag{1.8}$$

correspond to the steady state motions (1.1).

We next assume that the form of $L_2(q, q^\cdot, \xi^\cdot)$ is such as to be a strictly negative function of $q_i^\cdot, \xi_j^\cdot$ and that the function $L_0(q_i)$ is expandable in a Taylor series in the neighbourhood of (1.8).

By multiplying Eqs.(1.7) by $q_i^\cdot, \xi_j^\cdot$ and summing over all i, j, we obtain the energy equation

$$\frac{dH}{dt} = \sum_{i=1}^{k} Q_i q_i^\cdot + \sum_{j=k+1}^{n} \Phi_j \xi_j^\cdot \tag{1.9}$$

where the generalized energy

$$H(q, q^\cdot, \xi^\cdot) = L_2(q, q^\cdot, \xi^\cdot) - L_0(q) \tag{1.10}$$

Eq.(1.9) will be used when investigating the stability of the motions (1.8) when different assumptions are made concerning the generalized forces Q_i and Φ_j.

2. Let us first consider the case when the generalized forces are such that the quasicyclic velocities retain their specified constant values over the whole time of the motion /2, 4/

$$\varphi_j^\cdot = \omega_j, \text{ or } \xi_j^\cdot = 0 \tag{2.1}$$

whatever the values of q_i and $q_i^\cdot$ may be. It is obvious from the second group of Eqs.(1.7) that the equalities (2.1) will hold subject to the conditions

$$\Phi_j = \left(\frac{d}{dt}\frac{\partial L}{\partial \xi_j^\cdot}\right)_{\xi_j^\cdot = 0} \tag{2.2}$$

and the initial conditions $\xi_{j0}^\cdot = 0$. When these conditions are satisfied, the second group of Eqs.(1.7) is identically satisfied and the problem reduces to the investigation of just the first group of Eqs.(1.7) which can be treated as the equations of motion of a system with k degrees of freedom.

In this case Eq.(1.9) takes the form

$$\frac{dH^{(1)}}{dt} = \sum_{i=1}^{k} Q_i q_i^\cdot \quad (H^{(1)}(q, q^\cdot) = H(q, q^\cdot, 0) = L_2(q, q^\cdot, 0) - L_0(q) + L_0(q_0)) \tag{2.3}$$

Let us now consider the case when there are no generalized forces Q_i, i.e. $Q_i = 0$. In this case, the first k equations of (1.4) reduce to the equations

$$\partial L_0/\partial q_i = 0 \tag{2.4}$$

which signify that the function $L_0(q)$ has a stationary value of the motion (1.8).

On the basis of Lagrange's theorem concerning the stability of an equilibrium /5/ and its generalization to stability with respect to some of the variables /6/, we conclude that, if the function $L_0(q)$ has an isolated maximum in the case of the motion (1.8) (the function $L_0(q) - L_0(q_0)$ is strictly negative with respect to the variables q_r $(r = 1, \ldots, m \leqslant k)$), the motion (1.8) is stable with respect to the variables $q_i(q_r)$ and $q_i^{\cdot}$.

According to the application of Lagrange's theorem and to the Kelvin-Chetayev theorems /5/, we conclude that, if the function $L_0(q) - L_0(q_0)$ can take positive values in as small a neighbourhood of $q_i = q_{i0}$ as may be desired and the number of positive coefficients accompanying the squares of the variables in the quadratic part of the expansion of this function in a Taylor's series is odd, the motion (1.8) is unstable. Gyroscopic stabilization is possible when there is an even number of such coefficients. If dissipative forces which are derivatives of a function $f(q, q^{\cdot})$, whish is strictly negative, with respect to $q_i^{\cdot}$ act on the system in its perturbed motion in the neighbourhood of (1.8), the motion (1.8) which is stable under potential forces becomes asymptotically stable, an unstable motion remains unstable, and gyroscopic stabilization breaks down.

Let us now consider the case when dissipative forces which are derivatives of the strictly negative function

$$2f(q_i^{\cdot}, \varphi_j^{\cdot}) = \sum_{i,j=1}^{k} \beta_{ij} q_i^{\cdot} q_j^{\cdot} + 2\sum_{i=1}^{k} \sum_{j=k+1}^{n} \beta_{ij} q_i^{\cdot} \varphi_j^{\cdot} + \sum_{r,s=k+1}^{n} \beta_{rs} \varphi_r^{\cdot} \varphi_s^{\cdot} \tag{2.5}$$

with constants coefficients $\beta_{ij} = \beta_{ji}$ with respect to $q_i^{\cdot}$, $\varphi_j^{\cdot}$ act on the system and on any part of its motion. In this case conditions (2.1) are ensured by the application of the generalized forces

$$\Phi_j = \left(\frac{d}{dt}\frac{\partial L}{\partial \xi_j^{\cdot}}\right)_{\xi_j^{\cdot}=0} - \sum_{i=1}^{k} \beta_{ji} q_i^{\cdot} - \sum_{s=k+1}^{n} \beta_{js}\omega_s \tag{2.6}$$

The equations of motion (1.7) have solutions of the form (1.8) subject to the conditions

$$\frac{\partial L_0}{\partial q_i} + \sum_{j=k+1}^{n} \beta_{ij}\omega_j = 0 \tag{2.7}$$

It follows from Eqs.(2.7) that, when $\omega_j \neq 0$ in the case of the motion (1.8), the function $L_0(q)$ does not have a stationary value.

By putting $q_i = q_{i0} + x_i$ in the perturbed motion and expanding the function $L_0(q)$ in a power series in x_i

$$L_0(q) = L_0(q_0) + \sum_{i=1}^{k} \left(\frac{\partial L_0}{\partial q_i}\right)_0 x_i + \frac{1}{2} \sum_{i,j=1}^{k} \left(\frac{\partial^2 L_0}{\partial q_i \partial q_j}\right)_0 x_i x_j + \ldots \tag{2.8}$$

we reprsent Eq.(2.3), taking account of (2.7), in the form

$$\frac{dH^{(2)}}{dt} = \sum_{i,j=1}^{k} \beta_{ij} x_i^{\cdot} x_j^{\cdot} \tag{2.9}$$

$$\left(H^{(2)}(x_i, x_i^{\cdot}) = L_2(q_{i_0} + x_i, x_i^{\cdot}, 0) - \frac{1}{2} \sum_{i,j=1}^{k} \left(\frac{\partial^2 L_0}{\partial q_i \partial q_j}\right) x_i x_j + \ldots\right)$$

The dotted line denotes a set of terms of a higher order of smallness.

Using Eq.(2.9), we conclude on the basis of the Kelvin-Chetayev theorems /5/ that, if the second variation

$$\delta^2 L_0 = \frac{1}{2} \sum_{i,j=1}^{k} \left(\frac{\partial^2 L_0}{\partial q_i \partial q_j}\right)_0 x_i x_j$$

of the function $L_0(q)$ is strictly negative in the neighbourhood of the motion (1.8), the motion (1.8) is asymptotically stable with respect to the variables $q_i, q_i^{\cdot}$. If, however, $\delta^2 L_0$ can take positive values in any neighbourhood of (1.8) no matter how small it is, then this motion is unstable. In the case when the right-hand side of Eq.(2.9) is at all times solely of negative form, the set $\sum_{i,j} \beta_{ij} q_i^{\cdot} q_j^{\cdot} = 0$ does not contain complete motions apart from $q_i^{\cdot} = 0$, and $\delta^2 L_0$ is a strictly negative function, the motion (1.8) is asymptotically stable according to the Barbashin-Krasovskii theorem /7/. If, however, $\delta^2 L_0$ takes positive values then, according to the Krasovskii theorem /7/, the motion (1.8) is unstable.

3. Let us now drop the assumption that the forces (2.2) or (2.6) are applied to the system and that equalities (2.1) are satisfied and consider the motion of a system with quasi-cyclic coordinates under the action of potential forces, which are derivatives of the Lagrange function $L(q, q^{\cdot}, \xi^{\cdot})$, and the generalized forces

$$Q_i = \frac{\partial f}{\partial q_i^{\cdot}} + F_i, \quad \Phi_j = \frac{\partial f}{\partial \varphi_j^{\cdot}} + F_j \tag{3.1}$$

where $f(q^{\cdot}, \varphi^{\cdot})$ is the dissipative function (2.5) with complete dissipation and $F_\nu\ (\nu = 1, \ldots, n)$ are certain additional forces. Conditions (1.4) for the existence of solutions of the form of (1.8) now take the form

$$\frac{\partial L_0}{\partial q_i} + \sum_{j=k+1}^{n} \beta_{ij}\omega_j + F_i = 0, \quad \sum_{l=k+1}^{n} \beta_{jl}\omega_l + F_j = 0 \tag{3.2}$$

The case when $F_\nu = \text{const.}$ has been investigated previously in /3/. In this case, taking account of (3.1) and (3.2), Eq.(1.9) takes the form

$$\frac{dH}{dt} = 2f(q^{\cdot}, \xi^{\cdot}) - \sum_{i=1}^{k} \left(\frac{\partial L_0}{\partial q_i}\right)_0 q_i^{\cdot} \tag{3.3}$$

or, when account is taken of (2.8),

$$dH^{(3)}/dt = 2f(x_i^{\cdot}, \xi_j^{\cdot})$$
$$(H^{(3)}(x, x^{\cdot}, \xi^{\cdot}) = L_2(q_{i0} + x_i, x_i^{\cdot}, \xi_j^{\cdot}) - \delta^2 L_0 + \ldots)$$

Using Eq.(3.3), we conclude on the basis of the Kelvin-Chetayev theorems /5/ that the unperturbed motion (1.8) is asymptotically stable with respect to the variables $q_i, q_i^{\cdot}, \xi_j^{\cdot}$ provided that the second variation $\delta^2 L_0$ of the function $L_0(q)$ is strictly negative, but unstable if $\delta^2 L_0$ can take positive values in any neighbourhood of (1.8), no matter how small.

Let us now consider the case when the additional forces F_ν, which are continuous functions of $\varphi_j^{\cdot}$ and possess continuous partial derivatives of up to the second order inclusive

$$F_\nu = F_\nu(\varphi_{k+1}^{\cdot}, \ldots, \varphi_n^{\cdot}) \quad (\nu = 1, \ldots, n) \tag{3.4}$$

are such that Eqs.(3.2) have solutions (1.8).

Since, in the perturbed motion, we shall have

$$F_\nu(\varphi_{k+1}^{\cdot}, \ldots, \varphi_n^{\cdot}) = F_\nu(\omega_{k+1}, \ldots, \omega_n) + \sum_{j=k+1}^{n} \left(\frac{\partial F_\nu}{\partial \varphi_j^{\cdot}}\right)_0 \xi_j^{\cdot} + \ldots \tag{3.5}$$

after substitution of (1.6), Eq.(1.9), when account is taken of (3.1), (3.2), and (3.5), takes the form

$$\frac{dH^{(3)}}{dt} = 2f(x_i^{\cdot}, \xi_j^{\cdot}) + \sum_{i=1}^{k} \sum_{j=k+1}^{n} \left(\frac{\partial F_i}{\partial \varphi_j^{\cdot}}\right)_0 x_i^{\cdot} \xi_j^{\cdot} + \sum_{j,\, s=k+1}^{n} \left(\frac{\partial F_j}{\partial \varphi_s^{\cdot}}\right)_0 \xi_j^{\cdot} \xi_s^{\cdot} + \ldots \tag{3.6}$$

According to the Kelvin-Chatayev theorems /5/, if the right-hand side of Eq.(3.6) is a strictly negative function of $x_i^{\cdot}, \xi_j^{\cdot}$, the unperturbed motion (1.8) is asymptotically stable with respect to the variables $q_i, q_i^{\cdot}, \xi_j^{\cdot}$ in the case of a strictly negative second variation $\delta^2 L_0$, and unstable if $\delta^2 L_0$ can take positive values.

Finally, let us consider the case when there are no dissipative forces and the generalized forces acting on the system have the form (3.1) when $f = 0$, the forces F_ν are of the form of (3.4), and $F_i = 0$. In this case the first k equations of (3.2) take the form (2.4) and the last $n-k$ equations of (3.2) take the form

$$F_j(\omega_{k+1}, \ldots, \omega_n) = 0 \tag{3.7}$$

Subject to the condition that the Jacobian of system (3.7) differs from zero, this system enables one to find the values $\varphi_j^{\cdot} = \omega_j$, for which the motion (1.8) holds. Eq.(1.9) takes the form

$$\frac{dH^{(4)}}{dt} = \sum_{j,\, s=k+1}^{n} \left(\frac{\partial F_j}{\partial \varphi_s^{\cdot}}\right)_0 \xi_j^{\cdot} \xi_s^{\cdot} + \ldots \tag{3.8}$$

$$(H^{(4)}(x,\ x^{\cdot},\ \xi^{\cdot}) = L_2(q_{i0}+x_i,\ \ x_i^{\cdot},\ \ \xi_j^{\cdot}) - L_0(q_{i0}+x_i) + L_0(q_{i0}))$$

Consequently, if the right-hand side of Eq.(3.8) is strictly negative and the set $\xi_j^{\cdot} = 0$ does not contain any complete motions of the system, apart from (1.8), the motion (1.8) is asymptotically stable with respect to $q_i, q_i^{\cdot}, \xi_j^{\cdot}$ in the case when the function $L_0(q)$ has an isolated maximum but unstable if $L_0(q) - L_0(q_0)$ can take positive values in any neighbourhood of $q_i = q_{i0}$, no matter how small.

On comparing this result with the conclusion drawn in Sect.2 concerning the stability of the motion when there are no dissipative forces acting, we conclude that forces of the type of (3.4) and (3.7) stabilize the motion, which is steady in the case of an isolated maximum of the function $L_0(q)$ to an asymptotically stable motion subject to the conditions that the right-hand side of Eq.(3.8) is strictly negative and that there are no complete motions in the set $\xi_j^{\cdot} = 0$ apart from (1.8).

In the special case of additional forces of the form $F_j(\varphi_j^{\cdot})$, the conditions that the right-hand side of Eq.(3.8) should be strictly negative reduce to the obvious inequalities

$$(dF_j/d\varphi_j^{\cdot})_{\varphi_j^{\cdot}=\omega_j} < 0$$

Since the nature of the extremum of the function $L_0(q)$ is determined by its second variation $\delta^2 L_0$ in the majority of cases, the sufficient conditions for the stability of the motion (1.8) reduce in practice in all of the cases which have been considered above to the strict negativeness of $\delta^2 L_0$.

4. As an example, let us consider the problem of the stability of the steady-state motions of an absolutely solid rotor of mass m with a vertical axis set up in elasticated bearings which, in the general case, exert non-linear reactions. These bearings are rigidly clamped on to a fixed mounting. This problem, which is of great interest in machine construction, has served as the example in the investigations of many authors (see the bibliography in /1/). The results are also applicable to the case of the plane-parallel motion of a rotor rotating on an isotropic inertialess flexible shaft /5/.

Let us assume /1/ that the non-linear reactions of the bearings are reduced to a single equivalent reaction $mF(\rho)$ which is solely dependent on the radial displacement $\rho = O_1O$ of the axis O of the rotor and is directed along the straight line OO_1 to the point O_1 where the plane of the motion of the centre of mass C intersects the axis of the undeformed bearings and that $F(0) = 0$ and the derivatives $dF/d\rho > 0$ and $d^2F/d\rho^2$ are continuous within the limits of admissible deformations of the bearings. We shall take the point O_1 as the origin of a fixed system of coordinate axes O_1xyz with a vertical axis z. The eccentricity $e = OC$.

In the case of plane-parallel motion a free solid body has three degrees of freedom, and three independent variables are sufficient to define its position. Correspondingly, we shall have three equations of motion for the rotor which can be obtained from the laws of motion of the centre of mass (two equations) and the change in the angular momentum relative to the centre of mass (one equation) /8/ or from linear combinations of these equations. The Lagrange equations, the mechanical sense of which depends on the particular coordinate system which is chosen, turn out to be the most convenient in many cases. For example, if the polar coordinates r, θ of the centre of mass C and the angle χ between O_1C and the eccentricity $e = CO$ are taken as the coordinates of the rotor /5/ such that $\rho^2 = r^2 + e^2 - 2re\cos\chi$, the Lagrange function will be equal to

$$L = \frac{m}{2}\left[r^{\cdot 2} + r^2\theta^{\cdot 2} + k^2(\theta^{\cdot} + \chi^{\cdot})^2 - 2\int_0^{\rho} F(\rho)\,d\rho\right] \tag{4.1}$$

where k is the central radius of inertia.

When this is done, the Lagrangian equations of motion have the simplest form and they express, respectively, as can readily be seen, the theorems on the motion of the centre of mass (in a projection on the direction of the radius-vector r) and on the angular momentum realtive to the points O_1 and C.

In the variables ρ, φ, ψ, where ρ, φ are the polar coordinates of the point O and ψ is the angle between the x-axis and the straight line OC, the Lagrange function /1/ is

$$L = \frac{m}{2}\Big[\rho^{\cdot 2} + \rho^2\varphi^{\cdot 2} + 2e\psi^{\cdot}(\rho\varphi^{\cdot}\cos(\psi - \varphi) - \tag{4.2}$$

$$\rho^{\cdot}\sin(\psi - \varphi)) + k_0^2\psi^{\cdot 2} - 2\int_0^{\rho} F(\rho)\,d\rho\Big]$$

where k_0 is the radius of inertia for the point O. In these variables, the Lagrange equations express the theorem on the motion of the centre of mass (in a projection on the direction of the radius-vector ρ), a consequence of which is that

$$d/dt\,(\boldsymbol{\rho} \times m\mathbf{v}_c) - \mathbf{v}_0 \times m\mathbf{v}_c = 0 \tag{4.3}$$

and the theorem on the angular momentum relative to the point O

$$dG_0/dt + v_0 \times m v_c = 0 \tag{4.4}$$

where v_c and v_0 are the velocities of the points C and O and G_0 is the angular momentum realtive to the point O.

Not one of the coordinates ρ, φ, ψ is cyclic but, nevertheless, the equations of motion have the area integral

$$\partial L/\partial\varphi^{\cdot} + \partial L/\partial\psi^{\cdot} = m[\rho^2\varphi^{\cdot} + e(\varphi^{\cdot} + \psi^{\cdot})\rho\cos(\psi - \varphi) - e\rho^{\cdot}\sin(\psi - \varphi) + k_0^2\psi^{\cdot}] = \mathrm{const} \tag{4.5}$$

as a consequence of the obvious equality $\partial L/\partial\varphi + \partial L/\partial\psi = 0$. This integral (in vector form) also follows from Eqs.(4.3) and (4.4) and, by combining them, we obtain the integral

$$G_{O_1} = G_0 + \rho \times m v_c = \mathrm{const}$$

which is equivalent to (4.5). Here G_{O_1} is the angular momentum relative to the point O_1.

The variables φ, ψ occur in the function (4.2) solely in the form of their difference $\theta = \psi - \varphi$ and it is therefore natural to replace one of them by the variable whereupon the second of the variables φ, ψ becomes a cyclic coordinate. For instance, if ρ, θ, ψ are taken as the coordinates of the rotor, the coordinate ψ will be a cyclic coordinate and a first integral, analogous to (4.5) corresponds to it. At the same time, unlike Eq.(4.4), the Lagrange equation for the variable ψ expresses the theorem on the angular momentum relative to the point O_1. If the variables $\rho, \theta, \xi = \psi - \omega t$, are taken as the coordinates of the rotor, the function (4.2) takes the form

$$L = \frac{m}{2}\Big[\rho^{\cdot 2} + \rho^2(\xi^{\cdot} + \omega - \theta^{\cdot})^2 + 2e(\xi^{\cdot} + \omega)(\rho(\xi^{\cdot} + \omega - \theta^{\cdot}) \times \cos\theta - \rho^{\cdot}\sin\theta) + k_0^2(\xi^{\cdot} + \omega)^2 - 2\int_0^{\rho} F(\rho)\,d\rho\Big] \tag{4.6}$$

The Lagrange equation for the variable ξ leads to the area integral

$$\partial L/\partial\xi^{\cdot} = m[\rho^2(\xi^{\cdot} + \omega - \theta^{\cdot}) + e(\rho(2(\xi^{\cdot} + \omega) - \theta^{\cdot})\cos\theta - \rho^{\cdot}\sin\theta) + k_0^2(\xi^{\cdot} + \omega)] = mc = \mathrm{const} \tag{4.7}$$

which is equivalent to (4.5) while the equations for ρ and θ have the same meaning as when the equations are written in the variables ρ, φ, ψ.

5. Let us now consider the motion of the rotor when there are no dissipative forces but subject to the condition that the moment due to forces of the form of (2.2) is applied to its axis. This ensures the constancy of the inherent angular velocity of rotation $\psi^{\cdot} = \omega$ $(\xi^{\cdot} = 0)$, where ω is a specified constant. It is readily seen that the moment

$$M = meF(\rho)\sin\theta \tag{5.1}$$

ensures the required condition.

It can be seen from (4.6) that the function $L_0(q)$ for the rotor is equal to

$$L_0(\rho, \theta) = \frac{m}{2}\Big[(\rho^2 + 2e\rho\cos\theta)\omega^2 - 2\int_0^{\rho} F(\rho)\,d\rho\Big] \tag{5.2}$$

Equations of the form (2.4) have the solution

$$\rho = r, \quad \theta = \gamma \tag{5.3}$$

subject to the conditions that the constants r, γ satisfy the equations

$$\omega^2(r + e\cos\gamma) = F(r), \quad \sin\gamma = 0 \tag{5.4}$$

Eqs.(5.4) have been investigated in /1/. The values $\gamma = 0, \pi$ are found from the second equation of (5.4). The two forms of steady-state motion of the rotor, in which the points O_1, O and C lie on a single line which rotates around O_1 with an angular velocity ω, correspond to these values. When $\gamma = 0$, the point O lies between the points O_1 and C and, when $\gamma = \pi$, the point C lies between the points O_1 and O. The approximate form of the amplitude-frequency characteristic

$$\omega^2 = F(r)/(r \pm e) \tag{5.5}$$

is depicted in Fig.1 of /1/ and the skeletal curve

$$\omega = \varkappa(r), \quad \varkappa^2(r) = F(r)/r$$

divides the curve (5.5) into left-hand $(\gamma = 0)$ and right-hand $(\gamma = \pi)$ branches. The upper sign in (5.5) to (5.7) refers to the left-hand branch and the lower sign to the right-hand branch. Equations are derived from (5.4) which determine the coordinates of the point of bifurcation K at which the tangent to the curve (5.5) is parallel to the r axis

$$(r \pm e) F'(r) = F(r), \quad \omega^2 = F'(r) \tag{5.6}$$

The stability of the motion (5.3) has been investigated to a first approximation in /1/. We shall investigate it on the strength of the full equations. According to the results of paragraph 2, the conditions for the strict negativeness of the second variation of the function (5.2) for the solution (5.3)

$$F'(r) > \omega^2, \quad \mp mer\,\omega^2 < 0 \tag{5.7}$$

are sufficient conditions for the stability of the motion (5.3) of the rotor with respect to the variables $\rho, \theta, \rho^\cdot$ and $\theta^\cdot$.

Inequalities (5.7) can only be satisfied for the whole of the left-hand branch of (5.5) $(\gamma = 0)$ if there is no point of bifurcation K on it (the case of a rigid characteristic) or on the part of this branch from the origin of coordinates up to the point K, if there is a point of bifurcation on this branch (in the case of a soft characteristic). The second of inequalities (5.7) is not satisfied on the right-hand branch $(\gamma = \pi)$ as it has the opposite sign for all points of this branch and gyroscopic stabilization is possible for those points of the second branch for which the first of inequalities (5.7) also has the opposite sign. In fact, such stabilization occurs in the first approximation /1/.

If dissipative forces with complete dissipation which are derivatives of the strictly negative Rayleigh function $2f(x_1, x_2, x_1^\cdot, x_2^\cdot)$ with respect to $x_i^\cdot$ where $x_1 = \rho - r,\ x_2 = \theta - \gamma$, act on the rotor in its perturbed motion, the motion (5.3), which is stable under conditions (5.7), becomes asymptotically stable and the gyroscopic stabilization breaks down.

A resistive force which is proportional to the velocity of the radial displacement of the centre O of the rotor and arises due to the formation of the roller bearings also turns out to have a similar effect on the stability of the motion of the rotor /9/. In this case the dissipative function has the form $2f(\rho^\cdot) = -m\mu\rho^{\cdot 2},\ \mu > 0$ and, in order to ensure that the condition $\psi^\cdot = \omega$ is satisfied, the motor must develop a moment which differs from expression (5.1) by the addition of the term $me\mu\rho^\cdot \sin\theta$. At the same time the equations for the steady-state motions have the previous form (5.4) and the conditions for the strict negativeness of $\delta^2 L_0$ reduce to the inequalities (5.7). The set $f(\rho^\cdot) = 0$ is $\rho^\cdot = 0$, i.e. $\rho = \rho_0 = \text{const}$ and the equations of motion of the rotor take the form

$$\rho_0(\omega - \theta^\cdot)^2 + \omega^2 e\cos\theta = F(\rho_0), \quad \rho_0\theta^{\cdot\cdot} + \omega^2 e \sin\theta = 0 \tag{5.8}$$

In the general case when $\theta^\cdot \neq 0$ these equations are incompatible since, from the first equation, we have

$$\theta^\cdot = \omega \mp \frac{1}{\sqrt{\rho_0}} \sqrt{F(\rho_0) - \omega^2 e\cos\theta}$$

and, as the result of integration, the second yields

$$\theta^\cdot = \pm \frac{1}{\sqrt{\rho_0}} \sqrt{2\omega^2 e\cos\theta + C}$$

where C is a constant of integration. It follows from this that Eqs.(5.8) do not have solutions $\rho_0 = \text{const}$ when $\theta^\cdot \neq 0$. If, however, $\theta^\cdot = 0,\ \theta = \gamma,\ \rho = \rho_0$, the constants ρ_0 and γ must satisfy the equations

$$\omega^2(\rho_0 + e\cos\gamma) = F(\rho_0), \quad \sin\gamma = 0$$

which are identical to Eqs.(5.4). Consequently, the set $f(\rho^\cdot) = 0$ does not contain any complete perturbed motions apart from (5.3). On the basis of the Barbashin-Krasovskii theorem, we conclude that, subject to conditions (5.7), the motion (5.3) of the rotor under the action of a dissipative force $-m\mu\rho^\cdot$ becomes asymptotically stable with respect to $\rho, \theta, \rho^\cdot, \theta$. Should these conditions not be satisfied, when one or both of inequalities (5.7) have opposite signs, the motion (5.3) is unstable according to Krasovskii's theorem /7/.

Let us now drop the assumption that there is a constant characteristic angular velocity of rotation $\psi^\cdot$ and consider the case when the motor develops a moment $mM(\psi^\cdot)$ and there are no dissipative forces /1/. In this case Eqs.(2.4) have the form (5.4) and Eqs.(3.7) reduce to a single equation

$$M(\omega) = 0 \tag{5.9}$$

and, moreover, the right-hand side of the equation of the form of (3.8) will be equal to $(dM/d\psi^\cdot)_0 \xi^{\cdot 2}$. It is obvious that, when $\xi^\cdot = 0$, the equations of the perturbed motion of the rotor do not admit of complete motions apart from (5.3). Consequently, subject to the condition

$$(dM/d\psi^\cdot)_0 < 0 \tag{5.10}$$

the motion (5.3) of the rotor will be either asymptotically stable with respect to the variables $\rho, \theta, \rho^{\cdot}, \theta^{\cdot}, \xi^{\cdot}$, if the inequalities (5.7) are satisfied, or unstable, if one or both of the inequalities (5.7) have opposite signs.

We note that, subject to condition (5.10), a motor with a moment $M(\psi^{\cdot})$ produces an effect in the perturbed motion which is similar to a force which is dissipative with respect to $\xi^{\cdot}$. This is explained by the destabilizing effect /1/ of such a motor compared with a motor which ensures a constant velocity $\psi^{\cdot} = \omega$. Dissipative forces destroy gyroscopic stabilization.

In concluding, let us consider the case when the motor is switched off and there are no resistive forces /5, 8/. From the integral (4.7), we find

$$\xi^{\cdot} = -\omega + \frac{c + \rho(\rho + e\cos\theta)\theta^{\cdot} + e\rho^{\cdot}\sin\theta}{\rho^2 + 2e\rho\cos\theta + k_0^2}$$

and, using Routh's method of disregarding cyclic coordinates, we determine the changed potential energy

$$W(\rho, \theta) = \frac{mc^2}{2(\rho^2 + 2e\rho\cos\theta + k_0^2)} + m\int_0^\rho F(\rho)\,d\rho \tag{5.11}$$

The equations for the stationary motions of the rotor $\partial W/\partial\rho = 0$, $\partial W/\partial\theta = 0$ admit of the solution (5.3) subject to the conditions

$$-\frac{c^2(r + e\cos\gamma)}{(r^2 + 2er\cos\gamma + k_0^2)^2} + F(r) = 0, \quad \sin\gamma = 0$$

which, when the value of the constant integral (4.7) is

$$c = \omega(r^2 + 2er\cos\gamma + k_0^2)$$

take the form (5.4). The conditions for an isolated minimum of the function (5.11), transformed taking account of the last equality

$$F'(r) + \omega^2\frac{3r^2 \pm 6re - k_0^2 + 4e^2}{r^2 \pm 2re + k_0^2} > 0, \quad \pm\omega^2 er > 0 \tag{5.12}$$

are sufficient conditions for the stability of the stationary motion of the rotor which is being considered with respect to the variables $\rho, \theta, \rho^{\cdot}, \theta^{\cdot}, \xi^{\cdot}$. Conditions (5.12) can only be satisfied in the case of the first regime of stationary motion, when $\gamma = 0$. We note /2/ that, if conditions (5.7) are satisfied, conditions (5.12) are also satisfied.

The author thanks D.R. Merkin for discussing the paper and for valuable remarks.

REFERENCES

1. MERKIN D.R., On the stability of the stationary motions of the axis of a spinning rotor installed in non-linear bearings, PMM, 47, 3, 1983.
2. RUMYANTSEV V.V., On the stability of uniform rotations of mechanical systems. Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhanika i Mashinostroenie, 6, 1962.
3. POZHARITSKII G.K., On the stability of dissipative system, PMM, 21, 4, 1957.
4. STEPANOV S.YA., On the relationship of the stability conditions for three different regimes of cyclic motions in a system, in: Problems of Analytical Mechanics and Stability and Control Theories (Problemy analiticheskoi mekhaniki, teorii ustoichivosti i upravleniya), 2, Izd. Kazan. Aviats. Inst., Kazan', 1976.
5. CHETAYEV N.G., Stability of Motion (Ustoichivost' dvizheniya), Gostikhizd., Moscow-Leningrad, 1946.
6. RUMYANTSEV V.V., Some problems on the stability of motion with respect to some of the variables, in: Mechanics of a Continuous Medium and Related Problems of Analysis (Mekhanika sploshnykh sredy i rodstevennye problemy analiza), Nauka, Moscow, 1972.
7. KRASOVSKII N.N., Some Problems in the Theory of the Stability of Motion (Nekotorye zadachi teorii ustoichivosti dvizheniya), Fizmatgiz, Moscow, 1959.
8. ISHLINSKII A.YU., Mechanics of Relative Motion and Inertial Force (Mekhanika otnositel'nogo dvizheniya i sily inertsii), Nauka, Moscow, 1981.
9. MERKIN D.R., On the stability of the stationary motions of the axis of a spinning rotor, Dokl. Akad. Nauk SSSR, 257, 2, 1981.

Translated by E.L.S.